HomeWork 9

1. Solitonic solution of the Gross-Pitaevskii equation

The Gross-Pitaevksii equation was motivated in the last tutorial. For real wave functions, the one-dimensional GP equation reads

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + a\Psi(x) + b\Psi^3(x) = 0$$

a) By introducing a new set of dimensionless variables,

$$\begin{array}{ll} \Psi(x) = \Psi_0 f(x/\xi) & \xi = \sqrt{\frac{\hbar^2}{2ma}} \\ y = x/\xi & \Psi_0 = \sqrt{|a|/b}, \end{array}$$

rewrite a simple differential equation for $f(y)$.

b) Multiply the equation obtained in **a)** by f' (assuming that $f' \neq 0$ everywhere). You should recognize a *Newton-like* differential equation that can be solved by the first integral of motion (exactly like in classical mechanics).

Integrate this Newton equation with the following boundary conditions : $\Psi(0) = 0$ and $\Psi'(0)$ is finite (i.e. the boundary conditions for an infinite wall). Use the separation of variables between f and y.

Answer :

$$\Psi(x) = \Psi_0 \tanh(\frac{x}{\sqrt{2}\xi})$$

c) Plot $\Psi(x)$ and give a physical interpretation of ξ.

2. Elementary excitations in the Bose liquid

a) Linearize the Bogoliubov dispersion (see Homework 7) close to $k = 0$ and calculate the speed of sound in this regime (I already showed you how to do this). Comment.

b) What is the critical velocity for $k \neq 0$ excitation to propagate ? (cf lecture)

3. A simple model of the photoelectric effect

Consider, in $1D$, a particle of mass m, placed in a contact potential $V(x) = -\alpha\delta(x)$, where α is a real positive constant.

a) Prove briefly (standard quantum mechanics) that, in such a potential, there is a single bound state, of negative energy $E_0 = -m\alpha^2/2\hbar^2$, associated with a normalized wave function $\phi_0(x) = \sqrt{m\alpha/\hbar^2}e^{-m\alpha|x|/\hbar^2}$.

For each positive value of the energy $E = \hbar^2k^2/2m$, there are two stationary wave functions, corresponding, respectively, to an incident particle coming from the left or from the right. the expression for this eigenfunction is :

$$\chi_k(x) = \begin{cases} \frac{1}{\sqrt{2\pi}}\left(e^{ikx} - \frac{e^{-ikx}}{1+i\hbar^2k/m\alpha}\right) & \text{for } x < 0 \\ \frac{1}{\sqrt{2\pi}}\frac{i\hbar^2k/m\alpha}{1+i\hbar^2k/m\alpha}e^{ikx} & \text{for } x > 0 \end{cases}$$

b) Show that the $\chi_k(x)$ are orthonormal.

Hint: The following relation can be used:

$$\int_{-\infty}^{0} e^{iqx} dx = \int_{0}^{\infty} e^{-iqx} = \lim_{\epsilon \to 0} \frac{1}{\epsilon + iq}$$
$$= \pi\delta(0) - iP(\frac{1}{q}),$$

where P denotes the principal value. Calculate the density of states $\rho(E)$ for a positive energy E.

c) Calculate the matrix element $\langle \chi_k | X | \phi_0 \rangle$ of the position observable X between the bound state $|\phi_0\rangle$ and the positive energy state $|\chi_k\rangle$ whose wave function was given above.

d) The particle, assumed to be charged with charge q, interacts with an electric field oscillating at the angular frequency ω. The perturbation is:

$$W(t) = -q\mathcal{E}X \sin \omega t$$

where $\mathcal{E}$ is a constant.

The particle is initially in the bound state $|\phi_0\rangle$. Assume that $\hbar\omega > -E_0$. Calculate, using the Fermi's golden rule, the transition probability w per unit of time to an arbitrary positive energy state. How does w vary with ω and $\mathcal{E}$?

Note: We remind you the form of the density of transition probability, w, in the case of a sinusoidal perturbation (extension of the Fermi's golden rule). The perturbation V couples a state $|\phi_i\rangle$ to a continuum of states $|\beta_f, E_f\rangle$ with energies E_f close to $E_i + \hbar\omega$.

$$w = \frac{\pi}{2\hbar} |\langle \beta_f, E_f = E_i + \hbar\omega | V | \phi_i \rangle|^2 \rho(\beta_f, E_f = E_i + \hbar\omega)$$

ρ being of course the density of states at a given energy.

For derivation, and discussion of this important result, I invite the serious student to (re)read Ch. XIII of Cohen-Tannoudji et al. and Ch. 5 of Sakurai.